EFFECT OF VISCOSITY ON THE STABILITY OF AN ELECTRIC ARC

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§1. Assuming the plasma to be thermal and neglecting radiation and change of density due to electromagnetic forces, we write the equations for a steady electric arc burning in a cylindrical channel without a gas flow in dimensionless form as follows (see, for example, [1]):

$$(1/r)[r\lambda(T)T']' + E^2\sigma(T) = 0,$$
$$(1/r)(rH)' = \sigma(T),\ \rho T = 1, \tag{1.1}$$
$$c_p = c_p(T),\ J = E\sigma(T),\ H_\varphi = EH.$$

As the boundary conditions we can choose

$$T_{|r=0} = 1,\quad T'_{|r=0} = H_{|r=0} = 0 \tag{1.2}$$

for a given value of E = const.

The scales of temperature T, density ρ, thermal conductivity λ, electrical conductivity σ, and specific heat at constant pressure c_p are the values of the corresponding parameters on the axis of the channel T_m, ρ_m, λ_m, σ_m, and c_{pm}.

The radius of the channel is chosen as the scale of length, and as the scales of electric field strength E, inherent magnetic field strength H_φ, and current density J, we choose, respectively,

$$E_m = (1/R_m)\sqrt{\lambda_m T_m/\sigma_m},\ H_m = E_m\sigma_m R_m,\ J_m = E_m\sigma_m.$$

It should be noted that, in addition to Eqs. (1.2), $T_{|r=1} = T_R$, where T_R is the dimensionless temperature of the wall, can also be chosen as a boundary condition; T_R can be determined from the three boundary conditions for an equation of second order [the first equation of system (1.1)] with unknown constant E.

We write the linearized dimensionless equations for perturbations in the form

$$\operatorname{div}(\lambda\nabla\theta + (d\lambda/dT)\theta\nabla T) = \rho c_p(\partial\theta/\partial t + \mathbf{v}\cdot\nabla T) - \mathbf{E}\cdot\mathbf{j} - \mathbf{e}\cdot\mathbf{J},$$
$$\rho\partial\mathbf{v}/\partial t = -\nabla p + S(\mathbf{J}\times\mathbf{h} + \mathbf{j}\times\mathbf{H}) + (1/P)\operatorname{Div}\tau, \tag{1.3}$$
$$\partial g/\partial t + \operatorname{div}(\rho\mathbf{v}) = 0,\ gT + \theta\rho = 0,$$
$$\mathbf{j} = \sigma\mathbf{e} + \mathbf{E}(d\sigma/dT)\theta + R_1\sigma\mathbf{v}\times\mathbf{H},$$
$$\operatorname{rot}\mathbf{h} = \mathbf{j} - R_2\partial\mathbf{e}/\partial t,\ \operatorname{rot}\mathbf{e} = -R_1\partial\mathbf{h}/\partial t,\ \operatorname{div}\mathbf{h} = 0,$$

where **v**, p, θ, **h**, **e**, **j** and g are the perturbations of the velocity vector, pressure, temperature, magnetic field strength vector, electric field strength vector, current density vector, and plasma density, respectively, τ is the tensor of the viscous stresses; $S = \mu_e J_m H_m t_m^2/\rho_m R_m = \mu_e \times \sigma_m T_m c_{pm}^2 R_m^2/\lambda_m$ is the Stewart number: $\mu_e = 4\pi\cdot 10^{-7}$ kg·m/C² is the magnetic permeability; $t_m = \rho_m c_{pm} R_m^2/\lambda_m$ is the characteristic time; $P = \rho_m R_m^2/\mu_m t_m = \lambda_m c_{pm}\mu_m$ is the viscosity parameter; μ_m is the dynamic viscosity on the axis of the channel; and R_1 and R_2 are dimensionless parameters determined by the formulas

$$R_1 = \mu_e\sigma_m\lambda_m/\rho_m c_{pm},\quad R_2 = \varepsilon_e\lambda_m/\sigma_m\rho_m c_{pm}R_m^2$$

(ε_e is the dielectric permittivity).

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We shall estimate the orders of magnitude of the dimensionless parameters occurring in the system of equations (1.3). Assuming that $\mu_e \sim 10^{-6}$, $\sigma_m \sim 10^3$, $T_m \sim 10^4$, $\rho_m \sim 10^{-2}$, $\lambda_m \sim 1$, $c_{pm} \sim 10^3$, $R_m \sim 10^{-2}$, $\mu_m \sim 10^{-6}$, and $\varepsilon_e \sim 10^{-12}$ (values of the scaled quantities are given in SI units), we obtain $S \sim 10^3$, $P \sim 10$, $R_1 \sim 10^{-5}$, and $R_2 \sim 10^{-13}$. In the future, we shall neglect terms in the system of equations (1.3) containing the parameters R_1 and R_2.

Describing the system of equations (1.3) in projections, we shall find the solutions for the unknown functions in the following form:

$$\begin{gathered}\theta(t,\ r,\ \varphi,\ z) \to (\theta(r)/\lambda)G,\\ v_r(t,\ r,\ \varphi,\ z) \to v(r)G,\ h_\varphi(t,\ r,\ \varphi,\ z) \to Eh(r)G,\\ e_z(t,\ r,\ \varphi,\ z) \to Ee(r)G,\ p(t,\ r,\ \varphi,\ z) \to \mathrm{S}p(r)G,\\ v_z(t,\ r,\ \varphi,\ z) \to (w(r)/ik)G,\ h_r(t,\ r,\ \varphi,\ z) \to h_r(r)G,\\ h_z(t,\ r,\ \varphi,\ z) \to h_z(r)G,\ e_\varphi(t,\ r,\ \varphi,\ z) \to e_\varphi(r)G,\\ v_\varphi(t,\ r,\ \varphi,\ z) \to v_\varphi(r)G,\end{gathered} \tag{1.4}$$

where $G = \exp(\omega t + ikz)$, ω is the complex frequency (unknown eigenvalue), and k is the wave number, i.e., the problem concerning the stability of the electric arc relative to symmetrical perturbations is being studied. In formulas (1.4), the subscripts r, φ, and z denote the projections of vectors on the corresponding coordinate axes. Substituting expressions (1.4) into Eqs. (1.3), we obtain two independent systems of equations,

$$\begin{gathered}\rho\omega v_\varphi = -E^2\mathrm{S}\sigma h_r + \frac{1}{P}\left\{\frac{1}{r^2}\left[r^3\mu\left(\frac{v_\varphi}{r}\right)'\right]' - k^2\mu v_\varphi\right\},\\ ike_\varphi = 0,\quad \frac{1}{r}(rh_r)' = -ikh_z,\quad h_z' - ikh_r + \sigma e_\varphi = 0;\end{gathered} \tag{1.5}$$

$$\begin{gathered}p' = -\rho\omega v - E^2\mathrm{S}\left[H\left(\sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta\right) + \sigma h\right] +\\ + \frac{1}{P}\left\{\frac{2}{r}(r\mu v')' - \mu\left(\frac{2}{r^2} + k^2\right)v + \mu w' - \frac{2}{3}\left[\mu\left(\frac{1}{r}(rv)' + w\right)\right]'\right\},\\ \frac{1}{r}(r\mu w')' = P\rho\omega w - Pk^2p - P\mathrm{S}E^2k^2Hh + k^2\frac{1}{r}(r\mu v)' +\\ + \frac{4}{3}k^2\mu w - \frac{2}{3}k^2\mu\frac{1}{r}(rv)',\\ \frac{1}{r}(r\rho v)' = \frac{\omega\theta}{\lambda T^2} - \rho w,\\ \frac{1}{r}(r\theta')' = \left(\frac{c_p\rho\omega}{\lambda} + k^2\right)\theta + c_p\rho T'v - E^2\left(2\sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta\right),\\ \frac{1}{r}(rh)' = \sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta,\quad e' = \frac{k^2}{\sigma}h;\end{gathered} \tag{1.6}$$

here and in the future, the primes denote the derivative with respect to r.

For the last three equations of system (1.5), if we assume the boundedness of the functions at zero and at infinity, then generally only a trivial solution exists. It can also be verified that $\omega < 0$ for the first equation of system (1.5), i.e., the electric arc is stable relative to those perturbations related to the system of equations (1.5). Therefore, in the future we shall be concerned with the study of the system of equations (1.6).

Assuming the boundedness of the functions at zero, and also that the boundary of the channel is an impermeable, non-current-conducting surface with a constant temperature, we write the boundary conditions for the system of equations (1.6) in the form

$$\begin{aligned} v = w' = \theta' = h = 0 \qquad &\text{for}\quad r = 0,\\ v = w = \theta = h = 0 \qquad &\text{for}\quad r = 1.\end{aligned} \tag{1.7}$$

Thus, a boundary-value problem in eigenvalues has originated, i.e., a value of should be found for which there is a nontrivial solution of the problem (1.6), (1.7). Having determined ω, the stability can be judged from the real part: For $\omega_r = \mathrm{Re}(\omega) > 0$, the perturbations increase with time, while for $\omega_r < 0$ they attenuate. We note that, in the first place, the system of equations (1.6) allows a reduction of order [for example, v' can be determined from the third equation of system (1.6) and can be substituted into the first equation] and, secondly, the problem (1.1), (1.2), (1.6), (1.7) is symmetrical relative to E and k; also, if $\omega = \omega_r + i\omega_i$ is an eigenvalue of the problem (1.6), (1.7), then $\bar{\omega} = \omega_r - i\omega_i$ also is an eigenvalue.

It will be advantageous to construct the spectrum of the eigenvalues of the problem (1.1), (1.2), (1.6), (1.7) in the range of measurement of the parameters E, k, S, and P where this spectrum can be determined. Obviously, the most successful region for constructing the spectrum of the eigenvalues is the region where $E^2 \ll 1$ and $k^2 \ll 1$.

§2. We shall undertake the investigation of the stability of the electric arc for $k^2 \ll 1$. We shall find the solutions for the eigenfunctions and eigenvalues of the problem (1.6), (1.7) in the region of $k^2 \ll 1$ in the form of power series with respect to the small parameter k^2:

$$
\begin{gathered}
p = (1/Pk^2)(p_0 + k^2 p_1 + k^4 p_2 + \ldots), \\
v = (1/\rho)(v_0 + k^2 v_1 + \ldots),\ w = w_0 + k^2 w_1 + \ldots, \\
\theta = \theta_0 + k^2\theta_1 + \ldots,\ e = e_0 + k^2 e_1 + \ldots, \\
h = h_0 + k^2 h_1 + \ldots,\ \omega = \omega_0 + k^2\omega_1 + \ldots
\end{gathered} \tag{2.1}
$$

Substituting expansions (2.1) into problem (1.6), (1.7), after simple transformations in the zero approximation with respect to k^2, we obtain the following system of equations (the subscript zero is omitted):

$$
\begin{gathered}
p' = 0,\quad (1/r)(r\mu w')' = p + P\omega\rho w, \\
\frac{1}{r}(rv)' = \frac{\omega}{\lambda T^2}\theta - \rho w, \\
\frac{1}{r}(r\theta')' = \frac{c_p\rho\omega}{\lambda}\theta + c_p T' v - E^2\left(2\sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta\right), \\
\frac{1}{r}(rh)' = \sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta,\quad e' = 0
\end{gathered} \tag{2.2}
$$

with boundary conditions (1.7).

In the region of $E^2 \ll 1$, there are two versions of the construction of the expansion of the eigenfunctions and eigenvalues of problem (2.2), (1.7) with respect to the small parameter E^2:

version 1,

$$
\begin{gathered}
p = p_0 + E^2 p_1 + E^4 p_2 + \ldots,\ v = v_0 + E^2 v_1 + \ldots, \\
w = w_0 + E^2 w_1 + \ldots,\ \theta = \theta_0 + E^2\theta_1 + \ldots, \\
h = h_0 + E^2 h_1 + \ldots,\ e = e_0 + E^2 e_1 + \ldots, \\
\omega = \omega_0 + E^2\omega_1 + \ldots;
\end{gathered} \tag{2.3}
$$

version 2,

$$
\begin{gathered}
p = p_0 + E^2 p_1 + E^4 p_2 + \ldots,\ v = v_0 + E^2 v_1 + \ldots, \\
w = w_0 + E^2 w_1 + \ldots,\ \theta = E^2(\theta_0 + E^2\theta_1 + \ldots), \\
h = E^2(h_0 + E^2 h_1 + \ldots),\ e = E^2(e_0 + E^2 e_1 + \ldots), \\
\omega = (1/P)(\omega_0 + E^2\omega_1 + \ldots).
\end{gathered} \tag{2.4}
$$

Substituting expansions (2.3) and (2.4) into problem (2.2),(1.7), we obtain in the zero approximation with respect to E^2 (taking into account the expansion of the unperturbed solution with respect to the parameter E^2) two boundary-value problems in eigenvalues, from which two eigenvalue spectra can be determined. The first spectrum, related to expansion (2.3), is determined by the formula $\omega_0 = -\gamma_n^2$, while the second spectrum, related to expansion (2.4), is determined by the formula $\omega_0 = -\lambda_N^2$, where γ_n and λ_n are the positive roots of the equations $J_0(\gamma_n) = 0$ and $\lambda_n J_0(\lambda_n) - 2J_1(\lambda_n) = 0$ (J_0 and J_1 are Bessel functions), numbered in ascending order.

The eigenvalues of the problem (2.2),(1.7), determined for $E^2 \ll 1$ by the approximate formulas (2.3) and (2.4), were extended numerically from the region of small electric field strengths to the region of realistic strengths (i.e., those values of E for which the wall temperature is considerably less than the temperature on the axis) with simple dependences of μ, c_p, λ, and σ on the temperature,

$$\mu = c_p = \lambda = 1,\ \sigma = T. \tag{2.5}$$

and for different values of the viscosity parameter P.

It was found that the real part of the eigenvalues is everywhere less than zero. Then, for a fixed value of E = 2.35 (which corresponds to a wall temperature of $T_R = 0.03$), the eigenvalue spectrum was investigated when P was varied from 0 to ∞. It was found also in this case that the real part of the eigenvalues is everywhere negative, i.e., the electric arc is stable relative to perturbations with an arbitrarily small wave number. At the same time, as follows from [2], in the case when there is no viscosity (ideal fluid), instability relative to long-wave perturbations originates if E > 2.15.

§3. The eigenvalue spectrum of the problem (1.6), (1.7), taking account of Eqs. (2.5), (1.1), and (1.2), obtained in the region of $k^2 \ll 1$, was extended into the region of large wave numbers for E = 2.35 and for different values of P and S. It was found that if the product $PS > 4.2 \cdot 10^3$, then instability originates with an increase in wave number, but as $k \to \infty$, the instability disappears; i.e., the electric arc is stable relative to perturbations having both arbitrarily small and arbitrarily large wave numbers. Further, when considering the stability of the electric arc without taking account of viscosity, it was found under similar conditions that the arc is unstable over the whole range of wave numbers (see [3]).

In Fig. 1a, the neutral curves (curves separating the stable regions from the unstable regions) of $P_eS_e = P_eS_e(k)$ are drawn, while in Fig. 1b the curves of $\omega_e = \omega_e(k)$ (the subscript e denotes neutral parameters, i.e., parameters for which $\omega = i\omega_i = i\omega_e$) are drawn for different Stewart numbers (curves 1-3 are for $S = 10^4$, $2.6 \cdot 10^4$, and 10^5). The complex P_eS_e is plotted along the ordinate axis, since in the case $\omega = i\omega_i = i\omega_e = 0$ the criterion of stability is the product PS. For a fixed Stewart number, the neutral curve can be determined in the following way. Moving along branch I in the direction shown by the arrow, transfer to branch II (branches I and II of the neutral curves are constructed for the case when $\omega_e = 0$, and, therefore, they occur for any Stewart number) and continue up to the intersection with the curve corresponding to the required Stewart number; then a turn to the right is made and movement is continued along this curve. With the movement given in this way, the region of instability is found on the left-hand side of the neutral curve.

The neutral curves in the limiting cases of very small and very large Stewart numbers were calculated. For $S \ll 1$, the asymptotic expansion with respect to the small parameter S can be constructed:

$$\begin{gathered} p = S(p_0 + Sp_1 + S^2p_2 + \ldots),\ v = v_0 + Sv_1 + \ldots, \\ w = w_0 + Sw_1 + \ldots,\ \theta = \theta_0 + S\theta_1 + \ldots, \\ h = h_0 + Sh_1 + \ldots,\ e = e_0 + Se_1 + \ldots, \\ \omega_e = S(\omega_{e0} + S\omega_{e1} + \ldots),\ P_e = (1/S)(P_{e0} + SP_{e1} + \ldots). \end{gathered} \tag{3.1}$$

Substituting expansion (3.1) in the problem (1.6), (1.7), we obtain the following system of equations in the zero approximation for the neutral oscillations:

$$\begin{gathered} p' = -i\rho\omega_e v - E^2\left[H\left(\sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta\right) + \sigma h\right] + \\ + \frac{1}{P_e}\left\{\frac{2}{r}(r\mu v')' - \mu\left(\frac{2}{r^2} + k^2\right)v + \mu w' - \frac{2}{3}\left[\mu\left(\frac{1}{r}(rv)' + w\right)\right]'\right\}, \\ \frac{1}{r}(r\mu w')' = i\rho\omega_e P_e w - k^2P_e p - E^2P_e k^2 Hh + k^2\frac{1}{r}(r\mu v)' + \\ + \frac{4}{3}k^2\mu w - \frac{2}{3}k^2\mu\frac{1}{r}(rv)', \\ \frac{1}{r}(r\rho v)' = -\rho w, \\ \frac{1}{r}(r\theta')' = c_p\rho T'v + k^2\theta - E^2\left(2\sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta\right), \\ \frac{1}{r}(rh)' = \sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta,\quad e' = \frac{k^2}{\sigma}h \end{gathered} \tag{3.2}$$

(the subscript zero is omitted) with the boundary conditions (1.7). The neutral curves for the problem (3.2), (1.7), calculated over a wide range of variation of the wave number, are shown in Fig. 1a, b by curve 0.

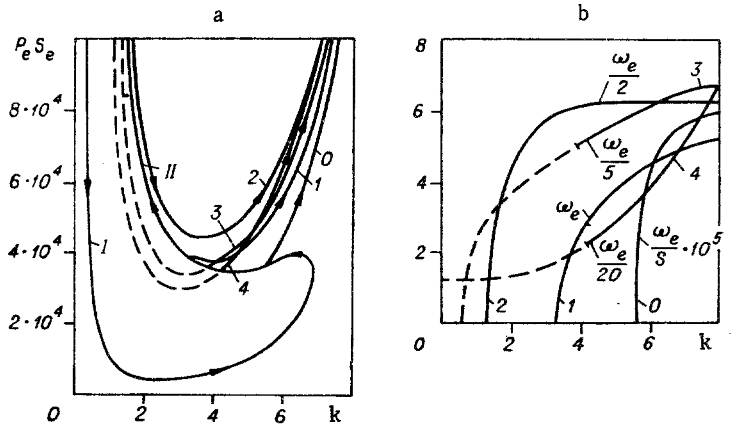

Fig. 1

In the region of S >> 1, the expansion of the eigenfunctions and eigenvalues of the problem (1.6), (1.7) with respect to the small parameter 1/S can also be constructed. For neutral oscillations, these expansions appear as follows:

$$\begin{gathered} p = S\left(p_0 + \frac{1}{S}p_1 + \frac{1}{S^2}p_2 + \ldots\right),\ v = v_0 + \frac{1}{S}v_1 + \ldots, \\ w = w_0 + \frac{1}{S}w_1 + \ldots,\quad \theta = \theta_0 + \frac{1}{S}\theta_1 + \ldots, \\ h = h_0 + \frac{1}{S}h_1 + \ldots,\quad e = e_0 + \frac{1}{S}e_1 + \ldots, \\ \omega_e = \omega_{e0} + \frac{1}{S}\omega_{e1} + \ldots,\quad P_e = \frac{1}{S}\left(P_{e0} + \frac{1}{S}P_{e1} + \ldots\right). \end{gathered} \tag{3.3}$$

Then substituting expression (3.3) into the problem (1.6), (1.7), we obtain in zero approximation with respect to 1/S (omitting the subscript zero) the system of equations

$$\begin{gathered} p' = -E^2\left[H\left(\sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta\right) + \sigma h\right] + \frac{1}{P_e}\left\{\frac{2}{r}(r\mu v')' - \right. \\ \left. -\mu\left(\frac{2}{r^2} + k^2\right)v + \mu w' - \frac{2}{3}\left[\mu\left(\frac{1}{r}(rv)' + w\right)\right]'\right\}, \\ \frac{1}{r}(r\mu w')' = -k^2P_ep - E^2P_ek^2Hh - k^2\frac{1}{r}(r\mu v)' + \\ + \frac{4}{3}k^2\mu w - \frac{2}{3}k^2\mu\frac{1}{r}(rv)', \\ \frac{1}{r}(r\rho v)' = -\rho w + \frac{i\omega_e}{\lambda T^2}\theta,\quad \frac{1}{r}(r\theta')' = \left(\frac{i\omega_e c_p\rho}{\lambda} + k^2\right)\theta + c_p\rho T'v \\ -E^2\left(2\sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta\right), \\ \frac{1}{r}(rh)' = \sigma e + \frac{1}{\lambda}\frac{d\sigma}{dT}\theta,\quad e' = \frac{k^2}{\sigma}h \end{gathered} \tag{3.4}$$

with the boundary conditions (1.7). For the problem (3.4), (1.7), the neutral parameters P_e and ω_e have been constructed as a function of the wave number. The results of the calculations are depicted in Fig. 1a, b (curve 4).

We shall call the values of the parameters P_eS_e and k satisfying the condition

$$P_cS_c = \min\,[P_eS_e(k)]$$

the critical values and we shall denote them by the subscript c. The critical parameters P_cS_c and k_c have been constructed as a function of the radius of the arc r_σ for a fixed wall temperature. In this case, λ, μ, and c_p, as before, were assumed to be equal to unity, and σ was assumed to be a piecewise-linear function of temperature:

$$\mu = \lambda = c_p = 1,$$

$$\sigma = \begin{cases} \dfrac{T - T_\sigma}{1 - T_\sigma} & \text{for}\quad T > T_\sigma\,(r < r_\sigma), \\ 0 & \text{for}\quad T < T_\sigma\,(r > r_\sigma), \end{cases}$$

where T_σ is the temperature below which the electrical conductivity is equal to zero.

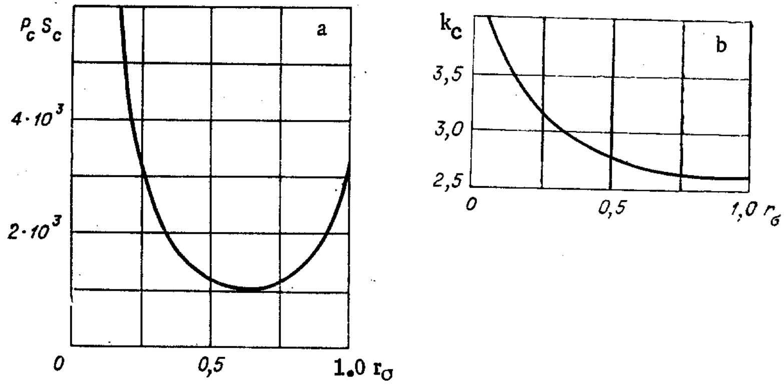

Fig. 2

Figure 2a shows the value of the critical complex as a function of the radius of the arc r_σ, and Fig. 2b shows the corresponding critical wave number k_c (the wall temperature in this case is equal to 0.03). As $r_\sigma \to 0$, the value of the complex $P_c S_c$ increases unrestrictedly, obviously, according to the law

$$P_c S_c = \text{const}/r_\sigma^2 (1 - T_\sigma)^2.$$

The critical curves for T_R = 0.015 and 0.06 were also constructed, but they differed from the curves shown in Fig. 2a, b by not more than 8%.

For the numerical calculation, the range of integration was divided into two regions: an electrically conducting region ($0 \le r \le r_\sigma$) and a nonelectrically conducting region ($r_\sigma \le r \le 1$); the solutions constructed in these two regions were joined when $r = r_\sigma$. The methods described in [4] were used in the calculations.

In conclusion, we note the following:

1) The viscosity proves to be a stabilizing influence (especially in the region of very small and very large wave numbers);

2) the criterion of stability is the product PS, the critical wave number varies only slightly (from 2.6 to 4.1) with a decrease in the radius of the arc from 1 to 0.01, and for this the phase velocity is equal to zero;

3) the model of an ideal liquid when investigating the problem of stability of an electric arc is applicable almost always, with the exception of a region of very small and very large wave numbers.

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